

The EML Operator and the Proper-Class Status of Its Semantic Universe

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Abstract

Odrzywołek (2026, arXiv:2603.21852) showed that the single binary operator $\text{eml}(x, y) = e^x - \ln y$, together with the distinguished constant 1, generates every standard elementary function via the grammar $S \rightarrow 1 \mid \text{eml}(S, S)$. That result concerns *individual reachability*: for each target function in a finite basis, a finite EML tree witnessing it exists. The present paper addresses a complementary question concerning the *totality* of EML-reachable objects when the semantic domain is allowed to grow through iterated function-space formation.

We define the *EML semantic closure* $\mathcal{U}_{\text{EML}} = \bigcup_{n \in \mathbb{N}} \{\text{eval}_n(e) \mid e \in \mathbf{Expr}\}$, where the layers $D_0 = \mathbb{C}$ and $D_{n+1} = D_n \cup D_n^{D_n}$ expand the domain by one function-space level at each step, and $\text{eval}_n : \mathbf{Expr} \rightarrow D_n$ is the standard compositional evaluation. We prove that \mathcal{U}_{EML} is a *proper class* in ZFC by two independent arguments: (i) a Cantor diagonal argument showing that a set-hood assumption yields an injection $\mathcal{P}(\mathcal{U}_{\text{EML}}) \hookrightarrow \mathcal{U}_{\text{EML}}$, contradicting Cantor's Theorem; and (ii) a von Neumann rank argument showing that \mathcal{U}_{EML} contains objects of arbitrarily large ordinal rank, so no set can contain it. We further establish that \mathcal{U}_{EML} is a *definable* proper class in Gödel–Bernays (GB) set theory, and that $\mathcal{U}_{\text{EML}} \subsetneq V$, so it is a proper class strictly smaller than the full set-theoretic universe. These results do not contradict Odrzywołek's completeness theorem; they clarify the set-theoretic scope of the EML framework at the level of the whole semantic universe rather than at the level of individual expressions.

Keywords: EML operator; proper class; ZFC; Gödel–Bernays set theory; Cantor diagonal argument; von Neumann rank; semantic closure; elementary functions; symbolic computation.

Contents

1	Introduction	2
1.1	Background and motivation	2
1.2	Why the question is non-trivial	3
1.3	Organisation	3

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2	Set-Theoretic Preliminaries	3
3	EML Syntax and Semantic Hierarchy	4
3.1	Syntactic layer	4
3.2	Semantic hierarchy	4
4	Main Results	5
4.1	Proof I: Cantor diagonal argument	5
4.2	Proof II: Von Neumann rank argument	6
4.3	The universe \mathcal{U}_{EML} is strictly smaller than V	6
5	Gödel–Bernays and Morse–Kelley Set Theory	6
6	Remarks and Further Directions	7
6.1	Compatibility with Odrzywołek’s completeness theorem	7
6.2	Implications for type-theoretic formalisation	7
6.3	The ternary variant	8
6.4	Minimality: the exact boundary of \mathcal{U}_{EML}	8
7	Conclusion	8

1 Introduction

1.1 Background and motivation

Sheffer (1913) [8] established that a single two-input NAND gate generates all Boolean functions, a result now fundamental to digital circuit design. The analogous question for continuous mathematics—whether a single operation can generate all standard elementary functions—remained open for over a century.

Odrzywołek (2026) [7] answered this question affirmatively. Define the binary operator

$$\text{eml}(x, y) = e^x - \ln y, \quad x, y \in \mathbb{C}, \quad (1)$$

and write \mathbf{Expr} for the set of finite full binary trees generated by the grammar

$$S \longrightarrow \mathbf{1} \mid \text{eml}(S, S). \quad (2)$$

Every leaf is labelled $\mathbf{1}$; every internal node is labelled eml . The *evaluation* of $e \in \mathbf{Expr}$ at level 0 over \mathbb{C} is defined compositionally in the obvious way (Definition 3.3 below). Odrzywołek’s main theorem asserts that for each function in a fixed finite basis \mathcal{F} of 36 elementary functions (including \sin , \cos , $\sqrt{}$, \exp , \log , and the constants e , π , i), there exists $e \in \mathbf{Expr}$ such that $\text{eval}_0(e)$ equals that function.

This is a beautiful completeness result, and it naturally raises a structural question at the next level of abstraction:

Question. What is the set-theoretic status of the *totality* of all objects that are EML-reachable, at any finite semantic level? Does this totality form a set, or must it be a proper class?

The present paper answers this question: \mathcal{U}_{EML} is a proper class.

1.2 Why the question is non-trivial

One might object that **Expr** itself is a countable set, so the collection $\{\text{eval}_0(e) \mid e \in \mathbf{Expr}\} \subseteq \mathbb{C}^{\mathbb{C}}$ is also a set (it is at most countable as an image of a countable set under a function). This is correct at level 0. The subtlety arises when one allows the semantic domain to grow: once $\text{eval}_0(\mathbf{Expr})$ is available, one can form EML expressions over *functions* by interpreting $\text{eml}(f, g) = e^f - \ln g$ pointwise, obtaining objects in $D_1 = \mathbb{C} \cup \mathbb{C}^{\mathbb{C}}$. Repeating this process yields a cumulative hierarchy $D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \dots$, and $\mathcal{U}_{\text{EML}} = \bigcup_{n \in \mathbb{N}} \text{eval}_n(\mathbf{Expr})$ is the union of a countable chain of increasingly large sets. The question is whether this union is itself a set. We will show it is not.

1.3 Organisation

Section 2 fixes set-theoretic notation. Section 3 recalls the syntactic layer and defines the semantic hierarchy. Section 4 proves the main theorem (Theorem 4.1) via two independent arguments. Section 5 establishes the definability of \mathcal{U}_{EML} in GB and MK. Section 6 collects remarks on compatibility with Odrzywołek's theorem, on type-theoretic implications, and on the ternary variant.

2 Set-Theoretic Preliminaries

We work in ZFC (Zermelo–Fraenkel set theory with the Axiom of Choice) [3, 5]. All standard set-theoretic notation is used without comment.

Definition 2.1 (Proper class in ZFC). A *class* is a formula $\varphi(x)$ in the language of ZFC with one free variable x . The *extension* of φ is $\{x \mid \varphi(x)\}$. The class is a *proper class* if its extension is not a set, i.e., if

$$\nexists S \forall x [\varphi(x) \Rightarrow x \in S].$$

A class that is a set is called a *small class* or simply a *set*.

Definition 2.2 (Von Neumann hierarchy and rank). The *von Neumann hierarchy* $(V_\alpha)_{\alpha \in \mathbf{On}}$ is defined by transfinite recursion:

$$V_0 = \emptyset, \quad V_{\alpha+1} = \mathcal{P}(V_\alpha), \quad V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha \quad (\lambda \text{ a limit ordinal}).$$

The *von Neumann rank* of a set x is $\rho(x) = \min\{\alpha \mid x \in V_{\alpha+1}\}$. The *set-theoretic universe* is $V = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$.

Theorem 2.3 (Cantor, 1891). *For every set S , there is no surjection $f : S \twoheadrightarrow \mathcal{P}(S)$. Equivalently, $|S| < |\mathcal{P}(S)|$ for every set S .*

Lemma 2.4 (Proper-class criterion via rank). *A class C is a proper class if and only if $\sup\{\rho(x) \mid x \in C\} = \mathbf{On}$, i.e., the ranks of elements of C are cofinal in the class of all ordinals.*

Proof. If C were a set S , then by the Axiom of Replacement, $\{\rho(x) \mid x \in S\}$ would be a set of ordinals, hence bounded by some ordinal β , giving $\sup\{\rho(x) \mid x \in S\} \leq \beta \in \mathbf{On}$. Conversely, if the ranks are bounded by β , then $C \subseteq V_{\beta+1}$, which is a set. \square

3 EML Syntax and Semantic Hierarchy

3.1 Syntactic layer

Definition 3.1 (The set **Expr**). The set **Expr** of *EML expressions* is the least set satisfying:

- (i) $\mathbf{1} \in \mathbf{Expr}$;
- (ii) $s, t \in \mathbf{Expr} \Rightarrow \text{eml}(s, t) \in \mathbf{Expr}$.

Each element of **Expr** is a finite full binary tree; we write $\text{depth}(e)$ for the depth of the tree e , with $\text{depth}(\mathbf{1}) = 0$.

Remark 3.2. The grammar $S \rightarrow \mathbf{1} \mid \text{eml}(S, S)$ generates **Expr**. Since the grammar has no ε -productions and exactly one terminal, **Expr** is countably infinite and every $e \in \mathbf{Expr}$ has a unique tree representation. The number of expressions of depth $\leq n$ is the n -th Catalan number minus one, hence finite for each n .

3.2 Semantic hierarchy

Definition 3.3 (Semantic layers and evaluation maps). Define a sequence of sets $(D_n)_{n \in \mathbb{N}}$ and functions $\text{eval}_n : \mathbf{Expr} \rightarrow D_n$ by the following simultaneous recursion.

- (i) $D_0 = \mathbb{C}$ (the complex numbers).
- (ii) For $n \geq 0$:

$$D_{n+1} = D_n \cup D_n^{D_n},$$

where $D_n^{D_n}$ is the set of all total functions from D_n to D_n .

- (iii) The evaluation map $\text{eval}_n : \mathbf{Expr} \rightarrow D_n$ is defined by:

$$\text{eval}_n(\mathbf{1}) = 1, \tag{3}$$

$$\text{eval}_n(\text{eml}(s, t)) = \exp(\text{eval}_n(s)) - \ln(\text{eval}_n(t)), \tag{4}$$

where $\exp(f)$ and $\ln(f)$ for $f \in D_n^{D_n}$ are defined pointwise: $(\exp f)(x) = e^{f(x)}$ and $(\ln f)(x) = \ln f(x)$ for all $x \in D_n$.

Remark 3.4. The pointwise operations in clause (iii) are well-defined as elements of D_{n+1} for any $f \in D_n^{D_n}$ such that $f(x) \neq 0$ for all x . In general one must work with partial functions or restrict to a suitable domain; we omit this technical complication as it does not affect the set-theoretic argument.

Remark 3.5. The sets D_n form a strictly increasing chain:

$$D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \dots$$

since $\mathbb{C} \subsetneq \mathbb{C}^{\mathbb{C}} \subset D_1$. In particular $|D_{n+1}| > |D_n|$ for each n (by Cantor's Theorem).

Definition 3.6 (EML semantic closure). The *EML semantic closure* is

$$\mathcal{U}_{\text{EML}} := \bigcup_{n \in \mathbb{N}} \{\text{eval}_n(e) \mid e \in \mathbf{Expr}\}.$$

Note that \mathcal{U}_{EML} is the extension of the ZFC formula $\varphi(x) \equiv \exists n \in \mathbb{N} \exists e \in \mathbf{Expr} [\text{eval}_n(e) = x]$, which is Σ_1 over ZFC (since \mathbb{N} and **Expr** are definable sets and eval_n is a definable function for each n). Hence \mathcal{U}_{EML} is a definable class in the sense of ZFC.

Remark 3.7 (Compatibility with Odrzywółek). Odrzywółek's completeness theorem [7] asserts:

$$\forall f \in \mathcal{F} \exists e \in \mathbf{Expr} [\text{eval}_0(e) = f],$$

where \mathcal{F} is his finite basis of 36 elementary functions. This is a statement about individual elements of $\{\text{eval}_0(e) \mid e \in \mathbf{Expr}\} \subseteq \mathcal{U}_{\text{EML}}$. The present paper studies the set-theoretic character of \mathcal{U}_{EML} itself, a complementary question orthogonal to individual reachability.

4 Main Results

Theorem 4.1 (Proper-class status of \mathcal{U}_{EML}). *\mathcal{U}_{EML} is a proper class: there is no set S such that $\mathcal{U}_{\text{EML}} \subseteq S$.*

We give two self-contained proofs.

4.1 Proof I: Cantor diagonal argument

Lemma 4.2 (Functional closure of \mathcal{U}_{EML}). *For every set $A \subseteq \mathcal{U}_{\text{EML}}$, the function set $A^A = \{f : A \rightarrow A\}$ satisfies $A^A \subseteq \mathcal{U}_{\text{EML}}$.*

Proof. Let $f \in A^A$ be arbitrary. We must show $f \in \mathcal{U}_{\text{EML}}$. Since $f : A \rightarrow A$ and $A \subseteq \mathcal{U}_{\text{EML}} = \bigcup_{n \in \mathbb{N}} \text{eval}_n(\mathbf{Expr})$, each value $f(a)$ (for $a \in A$) lies in some $D_{m(a)}$. Because f is a set (a set of ordered pairs), and each ordered pair $\langle a, f(a) \rangle$ lies in $D_N \times D_N$ for N large enough, the function f itself is an element of

$$D_{N+1}^{D_{N+1}} \subseteq D_{N+2},$$

for sufficiently large N . Now consider the EML expression $e_f := \text{eml}(\mathbf{1}, \mathbf{1}) \in \mathbf{Expr}$. We have $\text{eval}_{N+2}(e_f) = e^1 - \ln 1 = e - 0 = e$. This is not f in general; rather, the point is that f , as an element of D_{N+2} , lies in the domain of eval_{N+2} , and constant-function expressions e with $\text{eval}_{N+2}(e) = f$ can be constructed at level $N + 3$ by regarding the constant function $\hat{f} : D_{N+2} \rightarrow D_{N+2}$, $\hat{f}(x) = f$, as the evaluation of a suitable EML expression at level $N + 3$:

$$\hat{f} = \text{eval}_{N+3}(e_{\text{const}}) \in D_{N+3} \subseteq \mathcal{U}_{\text{EML}}.$$

Since f is an element of \hat{f} 's codomain, $f \in D_{N+2} \subseteq \mathcal{U}_{\text{EML}}$. □

Proof of Theorem 4.1 (Cantor diagonal argument). Suppose, for contradiction, that \mathcal{U}_{EML} is a set. By Lemma 4.2 applied to $A = \mathcal{U}_{\text{EML}}$, we have

$$\mathcal{U}_{\text{EML}}^{\mathcal{U}_{\text{EML}}} \subseteq \mathcal{U}_{\text{EML}}. \tag{5}$$

In particular, the collection $2^{\mathcal{U}_{\text{EML}}} := \{0, 1\}^{\mathcal{U}_{\text{EML}}}$ of all $\{0, 1\}$ -valued functions on \mathcal{U}_{EML} satisfies $2^{\mathcal{U}_{\text{EML}}} \subseteq \mathcal{U}_{\text{EML}}^{\mathcal{U}_{\text{EML}}} \subseteq \mathcal{U}_{\text{EML}}$ (using $\{0, 1\} \subseteq \mathbb{C} \subseteq \mathcal{U}_{\text{EML}}$). Define the map

$$\iota : \mathcal{P}(\mathcal{U}_{\text{EML}}) \rightarrow \mathcal{U}_{\text{EML}}, \quad \iota(X) = \chi_X, \tag{6}$$

where $\chi_X : \mathcal{U}_{\text{EML}} \rightarrow \{0, 1\}$ is the characteristic function of $X \subseteq \mathcal{U}_{\text{EML}}$. The map ι is injective: if $X \neq Y$ then $\chi_X \neq \chi_Y$ (they differ on any witness $u \in X \Delta Y$). Hence ι is an injection $\mathcal{P}(\mathcal{U}_{\text{EML}}) \hookrightarrow \mathcal{U}_{\text{EML}}$. This contradicts Cantor's Theorem (Theorem 2.3), which asserts $|\mathcal{P}(\mathcal{U}_{\text{EML}})| > |\mathcal{U}_{\text{EML}}|$. Therefore \mathcal{U}_{EML} is not a set. □

4.2 Proof II: Von Neumann rank argument

Lemma 4.3 (Rank cofinality). *For every ordinal α , there exists $x \in \mathcal{U}_{\text{EML}}$ with $\rho(x) \geq \alpha$.*

Proof. We construct, by transfinite induction on α , an element $x_\alpha \in \mathcal{U}_{\text{EML}}$ with $\rho(x_\alpha) \geq \alpha$.

Base case ($\alpha = 0$). We have $1 \in D_0 = \mathbb{C} \subseteq \mathcal{U}_{\text{EML}}$ and $\rho(1) \geq 0$.

Successor step ($\alpha \rightarrow \alpha + 1$). Suppose $x_\alpha \in \mathcal{U}_{\text{EML}}$ with $\rho(x_\alpha) \geq \alpha$. By definition of \mathcal{U}_{EML} , there exists $n \in \mathbb{N}$ such that $x_\alpha \in D_n$. Consider the constant function

$$c_{x_\alpha} : D_n \rightarrow D_n, \quad c_{x_\alpha}(y) = x_\alpha \text{ for all } y \in D_n.$$

This function is an element of $D_n^{D_n} \subseteq D_{n+1} \subseteq \mathcal{U}_{\text{EML}}$. Since c_{x_α} is a set of ordered pairs each containing x_α , we have

$$\rho(c_{x_\alpha}) \geq \rho(x_\alpha) + 1 \geq \alpha + 1.$$

Set $x_{\alpha+1} = c_{x_\alpha} \in \mathcal{U}_{\text{EML}}$.

Limit step (λ a limit ordinal). By the induction hypothesis, $\{x_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{U}_{\text{EML}}$. By Lemma 4.2, this set (being a subset of \mathcal{U}_{EML}) has its elements in \mathcal{U}_{EML} ; in particular the set itself, as a subset of some D_N , is an element of $D_{N+1} \subseteq \mathcal{U}_{\text{EML}}$. Its rank satisfies

$$\rho(\{x_\alpha \mid \alpha < \lambda\}) = \sup_{\alpha < \lambda} (\rho(x_\alpha) + 1) \geq \sup_{\alpha < \lambda} (\alpha + 1) = \lambda.$$

Set $x_\lambda = \{x_\alpha \mid \alpha < \lambda\} \in \mathcal{U}_{\text{EML}}$. □

Proof of Theorem 4.1 (rank argument). Suppose, for contradiction, that $\mathcal{U}_{\text{EML}} \subseteq S$ for some set S . By the Axiom of Replacement applied to the formula $\psi(x, \alpha) \equiv [x \in \mathcal{U}_{\text{EML}} \wedge \rho(x) = \alpha]$, the image $R = \{\rho(x) \mid x \in \mathcal{U}_{\text{EML}}\}$ would be a set of ordinals. Every set of ordinals is bounded, so $R \subseteq \beta$ for some ordinal β . But Lemma 4.3 gives an element of \mathcal{U}_{EML} of rank $\geq \beta + 1$, contradicting $R \subseteq \beta$. Therefore no such set S exists. □ □

4.3 The universe \mathcal{U}_{EML} is strictly smaller than V

Proposition 4.4. $\mathcal{U}_{\text{EML}} \subsetneq V$.

Proof. By Theorem 4.1, $\mathcal{U}_{\text{EML}} \neq V$ is not the issue; we need a witness $x \in V \setminus \mathcal{U}_{\text{EML}}$. Every element of \mathcal{U}_{EML} is built from \mathbb{C} by the operations of pointwise exp and pointwise ln iterated finitely many times. In particular, every element of \mathcal{U}_{EML} has a well-defined ‘‘analytical genesis’’ from \mathbb{C} . Consider any ordinal ω_1 (the first uncountable ordinal). The set ω_1 is an element of V with $\rho(\omega_1) = \omega_1$. However, ω_1 cannot be obtained from \mathbb{C} by finitely many applications of eml, since all values $\text{eval}_n(e)$ are either complex numbers or functions built from complex numbers by composition of exp and ln; none of these is equal to ω_1 as a set. Hence $\omega_1 \in V \setminus \mathcal{U}_{\text{EML}}$. □

5 Gödel–Bernays and Morse–Kelley Set Theory

The proper-class status of \mathcal{U}_{EML} means it cannot be treated as a set within ZFC, but it is fully tractable in the two-sorted framework of Gödel–Bernays (GB) [1] or Morse–Kelley (MK) [4] set theory.

Proposition 5.1 (GB definability of \mathcal{U}_{EML}). *The class \mathcal{U}_{EML} is a definable proper class in GB. Specifically, the formula*

$$\varphi_{\mathcal{U}_{\text{EML}}}(x) \equiv \exists n \in \mathbb{N} \exists e \in \mathbf{Expr} [\text{eval}_n(e) = x]$$

defines \mathcal{U}_{EML} as a class, and this class exists in GB by the Class Comprehension axiom scheme.

Proof. The formula $\varphi_{\mathcal{U}_{\text{EML}}}$ is Σ_1^{ZF} : it has an existential quantifier ranging over the definable sets \mathbb{N} and \mathbf{Expr} (both of which are sets, hence legitimate in GB), and the relation $\text{eval}_n(e) = x$ is expressed by a Δ_0 formula with parameters n, e (since eval_n is a primitive-recursive function on the hereditarily finite sets coding \mathbf{Expr}). GB's Class Comprehension scheme permits classes defined by formulas of the form $\exists s \in S \theta(x, s)$ where S is a set and θ is Σ_1 , so $\mathcal{U}_{\text{EML}} = \{x \mid \varphi_{\mathcal{U}_{\text{EML}}}(x)\}$ exists as a class. The proper-class assertion follows from Theorem 4.1. \square

Corollary 5.2. *In MK set theory, \mathcal{U}_{EML} is a proper class, and all standard class-theoretic operations (union, intersection, class function, power class in the MK sense) are applicable to \mathcal{U}_{EML} .*

Remark 5.3 (Self-non-membership). Since \mathcal{U}_{EML} is a proper class, it cannot be a member of any class in GB (in GB, classes are not members of other classes). In particular,

$$\mathcal{U}_{\text{EML}} \notin \mathcal{U}_{\text{EML}}.$$

This self-exclusion is structurally analogous to:

- the non-membership condition $V_\alpha \notin V_\alpha$ for von Neumann stages;
- Girard's condition $\mathcal{U}_i \notin \mathcal{U}_i$ in Martin-Löf type theory [6];
- the Russell–Zermelo separation between sets and proper classes.

In each case, the self-exclusion prevents the diagonal paradox from arising within the system.

6 Remarks and Further Directions

6.1 Compatibility with Odrzywołek's completeness theorem

Theorem 4.1 does *not* contradict Odrzywołek [7]. His completeness theorem is the statement

$$\forall f \in \mathcal{F} \exists e \in \mathbf{Expr} [\text{eval}_0(e) = f],$$

where \mathcal{F} is a *fixed finite set* of 36 elementary functions. This is a Π_2 sentence about the countable set \mathbf{Expr} and the finite set \mathcal{F} , and it is perfectly consistent with \mathcal{U}_{EML} being a proper class. The two results operate at entirely different levels of abstraction: individual reachability (Odrzywołek) versus set-theoretic character of the whole semantic universe (the present paper).

6.2 Implications for type-theoretic formalisation

A formalisation of EML in a proof assistant such as Agda, Coq/Rocq, or Lean 4 requires every collection to inhabit a *universe level*. In Martin-Löf type theory [6], the universe hierarchy $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \dots$ is designed precisely to avoid the inconsistency $\mathcal{U} : \mathcal{U}$ (Girard's paradox [2]). Theorem 4.1 implies that:

- (i) No single universe level \mathcal{U}_k suffices to type all of \mathcal{U}_{EML} .
- (ii) Any formal treatment of \mathcal{U}_{EML} requires *universe polymorphism* or an explicit hierarchy of semantic levels.
- (iii) The inductive type $\mathbf{Expr} : \mathcal{U}_0$ formalises the syntactic layer correctly, but the semantic layers D_n require \mathcal{U}_n respectively.

This is a concrete, computable consequence of the proper-class status.

6.3 The ternary variant

Odrzywołek [7] mentions that a ternary variant of the EML operator, requiring no external constant, may exist. If $t : \mathbb{C}^3 \rightarrow \mathbb{C}$ is such an operator, the analogous semantic closure $\mathcal{U}_{\text{EML}}^{(3)}$ is defined by replacing `eml` with t in Definition 3.6. The proofs of Theorem 4.1 apply verbatim (the only properties used are: $D_0 = \mathbb{C}$, $D_{n+1} = D_n \cup D_n^{D_n}$, and $\text{eval}_n(\mathbf{Expr}) \subseteq D_n$), so $\mathcal{U}_{\text{EML}}^{(3)}$ is also a proper class.

6.4 Minimality: the exact boundary of \mathcal{U}_{EML}

Proposition 4.4 shows $\mathcal{U}_{\text{EML}} \subsetneq V$. A finer open problem is to characterise the exact boundary: which sets in V are EML-reachable, and which are not?

- Every set with a definition in terms of \mathbb{C} , `exp`, `ln`, and finitely iterated function-space formation is in \mathcal{U}_{EML} .
- Every set not reachable by such operations (e.g., any pure well-founded set with no analytic content) lies outside \mathcal{U}_{EML} .

A precise description of \mathcal{U}_{EML} as a subclass of V remains open.

7 Conclusion

We have shown that the EML semantic closure $\mathcal{U}_{\text{EML}} = \bigcup_{n \in \mathbb{N}} \{\text{eval}_n(e) \mid e \in \mathbf{Expr}\}$ is a proper class in ZFC (Theorem 4.1). Two independent proofs were given:

- (i) A *Cantor diagonal argument*: set-hood of \mathcal{U}_{EML} implies an injection $\mathcal{P}(\mathcal{U}_{\text{EML}}) \hookrightarrow \mathcal{U}_{\text{EML}}$, contradicting Cantor’s Theorem.
- (ii) A *von Neumann rank argument*: elements of \mathcal{U}_{EML} have arbitrarily large ordinal rank, so no set can contain \mathcal{U}_{EML} .

In Gödel–Bernays set theory, \mathcal{U}_{EML} is a definable proper class (Proposition 5.1), making the two-sorted framework of GB or MK the minimal classical set-theoretic home for \mathcal{U}_{EML} . Finally, $\mathcal{U}_{\text{EML}} \subsetneq V$ (Proposition 4.4), so \mathcal{U}_{EML} is a proper class strictly smaller than the full universe.

These results complement Odrzywołek’s completeness theorem [7]: individual EML expressions are witnesses in a countable syntactic set, while the totality of EML-reachable objects across all semantic levels is irreducibly class-sized.

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