Unified EML Theory:
A Hierarchical Paraconsistent Fixed-Point Logic over ZFC

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Abstract

We introduce the *Unified EML System* $\mathcal{U}_{\mathrm{EML}}$, a formal framework combining classical ZFC/PA syntax, a transfinite semantic hierarchy of layers $\{L_\alpha\}_{\alpha\in\mathrm{Ord}}$, a paraconsistent valuation space modelled on First-Degree Entailment (FDE), and an evaluation map mediating between layers.

The framework rests on two structural novelties. First, a *Layer Isolation Principle* strictly separates the truth conditions of each layer from those of its successors, ensuring that $\mathcal{U}_{\mathrm{EML}}$ is a conservative extension of ZFC in its base layer: no paraconsistent glut or gap infects any formula of L_0. Second, the *Layer Transition Theorem* gives a precise account of how a Gödel sentence γ_n that is neither provable nor refutable within the n-th layer (a truth-value gap, $v_n(\gamma_n) = \emptyset$) acquires the paraconsistent truth value $v_{n+1}(\gamma_n) = \{T, F\}$ at the boundary with the successor layer—a *contradiction tolerance point* (CTP)—via the joint action of a reflection principle and the inherited internal provability predicate. The contradiction is not ad hoc: a Lawvere-style fixed-point argument shows it is the unique solution to the diagonal equation at the layer interface. In the next layer the Tarski jump resolves the glut, restoring a classical value $v_{n+1}(\mathrm{True}_n(\gamma_n)) = \{T\}$.

Iterating through all successor ordinals and taking unions at limit ordinals yields a transfinite open hierarchy $(D_\alpha)_{\alpha\in\mathrm{Ord}}$ that subsumes the systems of da Costa, Priest, and Belnap–Dunn as special cases, and that is gapless at every limit stage. All central claims are stated as formal definitions, axioms, and theorems with complete proofs; philosophical interpretation is confined to remarks.

Contents

1 Introduction **2**

2 Preliminaries **2**
2.1 First-Degree Entailment and Related Paraconsistent Logics 2
2.2 Lawvere's Fixed-Point Lemma 3
2.3 Gödel's Incompleteness Theorems 3

3 The Unified EML System **3**
3.1 Four-Component Definition 3
3.2 Layer Domains 4
3.3 Core Axioms 4

4 The Paraconsistent Evaluation Function **5**
4.1 Internal and External Truth Predicates 5
4.2 Layer-Relative Truth 5
4.3 Contradiction Tolerance Points 6

5 Layer Transition Theorem **6**
5.1 Gödel Sentences as Transition Drivers 6
5.2 The Transition Mechanism: Provability vs. Truth 6
5.3 Fixed-Point Structure via Lawvere 7

6 Conservative Extension over ZFC **8**

7 Transfinite Extension **8**
7.1 Extension Through All Ordinals 8
7.2 The Open Hierarchy 9

8 Correspondence with Known Systems **9**

9 Main Theorem **10**

10 Philosophical Remarks **10**

1 Introduction

A recurrent theme in the foundations of mathematics is that no sufficiently expressive formal system can be closed under its own semantic reflection. Gödel's incompleteness theorems [4], Tarski's undefinability of truth [9], and Lawvere's categorical fixed-point lemma [5] each exhibit, from a different angle, the same structural phenomenon: *a layer of expressive power generates objects that escape that layer.*

The present paper proposes a unified framework—the *Extensional Metalogical Layer* (EML) system—that treats this layer-transcendence as the primary object of study rather than a limit to be avoided. The framework has two previous incarnations. An earlier version established the four-component definition of $\mathcal{U}_{\text{EML}}$, the FDE valuation space, and the rough shape of the layer hierarchy but left implicit the boundary between the internal provability predicate and the external truth predicate, creating an apparent circularity in the transition argument. A revised version introduced the Layer Isolation Principle to address this gap but presented only a sketch of the transfinite extension.

The present paper merges and completes both accounts. Section 2 recalls the background logics (FDE, LP, da Costa's C_n) and the Lawvere fixed-point lemma. Section 3 gives the complete four-component definition together with the Layer Isolation Principle. Section 4 constructs the paraconsistent evaluation function. Section 5 proves the Layer Transition Theorem in its definitive form, making the separation of provability and truth explicit. Section 6 establishes conservative extension over ZFC. Section 7 extends the construction to all ordinals. Section 8 locates $\mathcal{U}_{\text{EML}}$ within the landscape of known logics. Section 9 states and proves the Unified EML Main Theorem. A concluding philosophical remark is offered in Section 10.

Notational conventions. We write $\vdash$ for the provability relation of the theory under discussion, $\nvdash$ for its negation, and $\ulcorner\phi\urcorner$ for the Gödel code of a formula ϕ. The FDE truth-value set is $\mathcal{P}(\{T, F\}) = \{\emptyset, \{T\}, \{F\}, \{T, F\}\}$.

2 Preliminaries

2.1 First-Degree Entailment and Related Paraconsistent Logics

Definition 2.1 (FDE Valuation)**.** A *First-Degree Entailment (FDE) valuation* is a function $\mu : \text{Atom} \to \mathcal{P}(\{T, F\})$ extended to all formulas by:

$$\begin{aligned} T \in \mu(\neg\phi) &\iff F \in \mu(\phi), \\ F \in \mu(\neg\phi) &\iff T \in \mu(\phi), \\ T \in \mu(\phi \wedge \psi) &\iff T \in \mu(\phi) \text{ and } T \in \mu(\psi), \\ F \in \mu(\phi \wedge \psi) &\iff F \in \mu(\phi) \text{ or } F \in \mu(\psi), \\ T \in \mu(\phi \vee \psi) &\iff T \in \mu(\phi) \text{ or } T \in \mu(\psi), \\ F \in \mu(\phi \vee \psi) &\iff F \in \mu(\phi) \text{ and } F \in \mu(\psi). \end{aligned}$$

The four possible values are $\emptyset$ (neither true nor false), $\{T\}$ (true only), $\{F\}$ (false only), and $\{T, F\}$ (both—a *glut*).

Definition 2.2 (LP Valuation)**.** *Priest's Logic of Paradox* (**LP**) restricts FDE by requiring $\mu(\phi) \neq \emptyset$ for every formula ϕ (no gaps), while permitting gluts.

Definition 2.3 (Non-Explosion)**.** A valuation μ satisfies *non-explosion* (paraconsistency) if there exist formulas ϕ, ψ such that $\mu(\phi) = \{T, F\}$ and $T \notin \mu(\psi)$, i.e., $\phi, \neg\phi \not\models \psi$ for some ψ.

The systems $\{C_n : n \geq 1\}$ of da Costa [3] satisfy Definition 2.3 through a hierarchy of consistency operators $\circ_n$; **LP** satisfies it via the four-valued semantics of Definition 2.2; both systems are subsumed by $\mathcal{U}_{\text{EML}}$ as shown in Section 8.

2.2 Lawvere's Fixed-Point Lemma

The following abstract form of the diagonal argument underlies our construction.

Theorem 2.4 (Lawvere Fixed-Point Lemma [5]). *Let* $\mathbf{C}$ *be a cartesian closed category and* $f : A \to Y^A$ *a morphism. If* f *is a point-surjection (every* $g : \mathbf{1} \to Y^A$ *lies in the image of* f *up to generalised elements), then every endomorphism* $t : Y \to Y$ *has a fixed point* $y : \mathbf{1} \to Y$ *with* $t \circ y = y$.

We apply Theorem 2.4 in the category **Set** with $Y = \mathcal{P}(\{T, F\})$ and $A = D_n$, where the point-surjection is provided by the Gödel numbering $\ulcorner\cdot\urcorner : D_n \to D_n$.

2.3 Gödel's Incompleteness Theorems

Let T be a consistent, recursively axiomatised extension of PA.

Theorem 2.5 (Gödel First Incompleteness [4]). *There exists a sentence* $\gamma_T \in \mathcal{L}_T$ *such that* $T \nvdash \gamma_T$ *and* $T \nvdash \neg\gamma_T$.

Theorem 2.6 (Tarski Undefinability [9]). *No consistent extension* T *of* PA *can define its own truth predicate within* $\mathcal{L}_T$.

These theorems are applied layer-by-layer throughout the paper: T is replaced by the theory of D_n, and γ_T becomes the transition sentence γ_n driving the generation of L_{n+1}.

3 The Unified EML System

3.1 Four-Component Definition

Definition 3.1 (Unified EML System). The *Unified EML System* is the four-tuple

$$\mathcal{U}_{\mathrm{EML}} = (L_1,\ L_2,\ v,\ \alpha),$$

where:

(i) $L_1 = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$ is the *EML semantic hierarchy*, a transfinite family of sublanguages with L_0 the classical first-order language of $\mathrm{ZFC} \cup \mathrm{PA}$ and $L_\alpha \subsetneq L_\beta$ for $\alpha < \beta$;

(ii) $L_2 = \mathcal{P}(\{T, F\})$ is the *paraconsistent valuation space*, the power set $\{\emptyset, \{T\}, \{F\}, \{T, F\}\}$ equipped with the FDE partial order $\emptyset \le \{T\} \le \{T, F\}$, $\emptyset \le \{F\} \le \{T, F\}$;

(iii) $v : L_1 \to L_2$ is the *evaluation map* (defined in Section 4);

(iv) $\alpha : L_1 \to \mathrm{Ord}$ assigns to each formula ϕ its *layer of origin* $\alpha(\phi) := \min\{\beta : \phi \in L_\beta\}$.

3.2 Layer Domains

Definition 3.2 (Layer Domain)**.** The *layer domain* D_α is defined by transfinite recursion:

$$D_0 := \text{the standard model of ZFC (the von Neumann universe } V), \quad (1)$$

$$D_{\alpha+1} := D_\alpha \cup D_\alpha^{D_\alpha}, \quad (2)$$

$$D_\lambda := \bigcup_{\alpha<\lambda} D_\alpha \quad \text{for limit ordinals } \lambda. \quad (3)$$

Here $D_\alpha^{D_\alpha}$ denotes the set of all functions $D_\alpha \to D_\alpha$.

Remark 3.3. Equation (2) mirrors the set-theoretic power-set construction at the level of function spaces, analogous to the universe hierarchy in Girard's System F and to the cumulative hierarchy $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.

3.3 Core Axioms

Axiom 1 (Layer Isolation Principle)**.** The valuation of any formula is fixed at its layer of origin:

$$v(\phi) = v_{\alpha(\phi)}(\phi).$$

Higher-layer predicates (e.g., $\mathrm{True}_{\alpha+1}$) cannot retroactively alter the truth value of sentences in L_α.

Axiom 2 (ZFC Base Layer)**.** The base layer (L_0, v_0) is the standard model of ZFC with Tarskian semantics. For all $\phi \in L_0$, $v(\phi) \in \{\{T\}, \{F\}\}$.

Axiom 3 (Layer Separation)**.** $L_0 \cap L_2 = \emptyset$: the classical base language and the paraconsistent valuation space share no syntactic objects.

Axiom 4 (Non-Explosion)**.** The evaluation map v satisfies Definition 2.3: there exist $x \in L_1$ with $v(x) = \{T, F\}$, yet $v(y) \neq \{T, F\}$ for some $y \in L_1$. In particular, $v(x) = \{T, F\}$ does not imply $v(y) = \{T, F\}$ for all y.

Axiom 5 (Hierarchical Generation)**.** $D_{\alpha+1} = D_\alpha \cup D_\alpha^{D_\alpha}$ (cf. Definition 3.2, Eq. (2)).

4 The Paraconsistent Evaluation Function

4.1 Internal and External Truth Predicates

A central contribution of the present paper is the rigorous separation of two distinct notions of truth, one internal to each layer and one external to it.

Definition 4.1 (Internal Provability Predicate)**.** For each ordinal α, let $\mathrm{Prov}_\alpha(\cdot)$ be the standard arithmetical provability predicate for the theory $\mathrm{Th}(D_\alpha)$ formalised in L_α. By Theorem 2.6, Prov_α is expressible in L_α but does not coincide with truth in D_α.

Definition 4.2 (External Truth Predicate / Tarski Jump)**.** For each ordinal α, the *Tarski jump* is the truth predicate $\mathrm{True}_\alpha : D_\alpha \to \mathcal{P}(\{T, F\})$ formalised in $L_{\alpha+1}$, defined by

$$\mathrm{True}_\alpha(\phi) = \{T\} \iff D_\alpha \models \phi.$$

By Theorem 2.6, $\mathrm{True}_\alpha \notin L_\alpha$, hence $\mathrm{True}_\alpha \in L_{\alpha+1} \setminus L_\alpha$.

The Layer Isolation Principle (Axiom 1) ensures that True_α is available only from $L_{\alpha+1}$ onward; it cannot be used to evaluate formulas *within* L_α. This is the key structural fix relative to earlier versions of EML.

4.2 Layer-Relative Truth

For each ordinal α, let $\mathrm{Th}(D_\alpha)$ denote the first-order theory of D_α in the language L_α.

Definition 4.3 (Layer-Relative Valuation)**.** Define $v_\alpha : L_\alpha \to \mathcal{P}(\{T, F\})$ by:

$$v_\alpha(\phi) := \begin{cases} \{T\} & \text{if } \mathrm{Th}(D_\alpha) \vdash \phi \text{ and } \mathrm{Th}(D_\alpha) \nvdash \neg\phi, \\ \{F\} & \text{if } \mathrm{Th}(D_\alpha) \nvdash \phi \text{ and } \mathrm{Th}(D_\alpha) \vdash \neg\phi, \\ \{T, F\} & \text{if } \mathrm{Th}(D_\alpha) \vdash \phi \text{ and } \mathrm{Th}(D_\alpha) \vdash \neg\phi, \\ \emptyset & \text{if } \mathrm{Th}(D_\alpha) \nvdash \phi \text{ and } \mathrm{Th}(D_\alpha) \nvdash \neg\phi. \end{cases} \tag{4}$$

Definition 4.4 (Global Evaluation Map)**.** The global evaluation map $v : L_1 \to \mathcal{P}(\{T, F\})$ is defined by

$$v(\phi) := v_{\alpha(\phi)}(\phi),$$

where $\alpha(\phi) = \min\{\beta : \phi \in L_\beta\}$ is the layer of origin of ϕ.

Note that Definition 4.4 is precisely Axiom 1 instantiated to the layer-relative valuations; the two together guarantee that v is well-defined.

Proposition 4.5 (Well-Definedness of v)**.** *v is well-defined and satisfies the FDE clause laws of Definition 2.1.*

Proof. Well-definedness follows from the strict hierarchy $L_0 \subsetneq L_1 \subsetneq \cdots$ and Axiom 1, which makes the layer-of-origin map $\alpha(\cdot)$ single-valued. The four cases of (4) are exhaustive and mutually exclusive by the independence of the predicates $\vdash \phi$ and $\vdash \neg\phi$ (each is a syntactic condition on $\mathrm{Th}(D_\alpha)$). The FDE clause laws are inherited from the underlying classical proof system of each $\mathrm{Th}(D_\alpha)$; the paraconsistent behaviour arises only in the $\{T, F\}$ branch, which by Axiom 4 does not propagate to all formulas. □

4.3 Contradiction Tolerance Points

Definition 4.6 (Contradiction Tolerance Point)**.** A formula $\phi \in L_1$ is a *contradiction tolerance point (CTP)* at layer α if $v_\alpha(\phi) = \{T, F\}$. The set of all CTPs at layer α is

$$\mathrm{CTP}(\alpha) := \{\phi \in L_\alpha : v_\alpha(\phi) = \{T, F\}\}.$$

Proposition 4.7 (CTPs are non-empty at every transition boundary). *For each* $n \geq 0$, $\mathrm{CTP}(n+1) \neq \emptyset$.

Proof. The Gödel sentence γ_n of D_n (constructed in Section 5) satisfies $v_{n+1}(\gamma_n) = \{T, F\}$ by the Layer Transition Theorem (Theorem 5.3). □

5 Layer Transition Theorem

5.1 Gödel Sentences as Transition Drivers

Definition 5.1 (Gödel Sentence of Layer n). For $n \geq 0$, let $\gamma_n \in L_n$ be the Gödel sentence of $\mathrm{Th}(D_n)$, i.e., the sentence satisfying

$$\mathrm{Th}(D_n) \vdash \gamma_n \leftrightarrow \neg\mathrm{Prov}_n(\ulcorner \gamma_n \urcorner)\,.$$

By Theorem 2.5, $\mathrm{Th}(D_n) \nvdash \gamma_n$ and $\mathrm{Th}(D_n) \nvdash \neg\gamma_n$, so $v_n(\gamma_n) = \emptyset$.

Remark 5.2. The value $\emptyset$ (neither true nor false within L_n) signals *layer inadequacy*: γ_n is expressible in L_n but receives no definite truth value there. This FDE truth-value gap is precisely what the layer transition fills. The gap arises because γ_n encodes a statement about the *provability predicate* Prov_n, which is internal to L_n; the truth predicate True_n, which is the correct semantic counterpart, is only available from L_{n+1} onward.

5.2 The Transition Mechanism: Provability vs. Truth

The core of the Layer Transition Theorem is the interaction between Prov_n (internal to L_n) and True_n (available only in L_{n+1}).

Theorem 5.3 (Layer Transition Theorem). *For each* $n \geq 0$*:*

(i) $v_n(\gamma_n) = \emptyset$ *(gap in* L_n*).*

(ii) $v_{n+1}(\gamma_n) = \{T, F\}$ *(glut at the boundary* L_n/L_{n+1}*).*

(iii) $v_{n+1}(\mathrm{True}_n(\gamma_n)) = \{T\}$ *(the Tarski jump resolves the glut in* L_{n+1}*).*

Proof. *(i)* Immediate from Definition 5.1 and Theorem 2.5.

(ii) We must show that both γ_n and $\neg\gamma_n$ are provable in $\mathrm{Th}(D_{n+1})$.

Positive side. In D_{n+1}, the domain D_n is a set-theoretic object (by Definition 3.2). The standard reflection principle for D_n is expressible and provable in $\mathrm{Th}(D_{n+1})$: if $\mathrm{Th}(D_n) \nvdash \gamma_n$, then in the standard model, γ_n is genuinely unprovable in D_n, which is exactly what γ_n asserts. Hence $D_{n+1} \models \gamma_n$, so $\mathrm{Th}(D_{n+1}) \vdash \gamma_n$.

Negative side. The sentence $\neg\gamma_n$ is equivalent (by the Gödel fixed-point equivalence) to $\mathrm{Prov}_n(\ulcorner \gamma_n \urcorner)$, the assertion that γ_n is provable in D_n. In D_{n+1}, the provability predicate Prov_n of D_n is a *purely syntactic* object inherited from L_n. The ω-inconsistency argument applies: D_{n+1} can verify, for each finite k, that the k-th derivation in D_n does not prove γ_n; but D_{n+1} also has access to the non-standard proof codes that satisfy $\mathrm{Prov}_n(\ulcorner \gamma_n \urcorner)$ in the non-standard part of its arithmetic. In the paraconsistent boundary logic of the layer transition, both the semantic (reflection) derivation and the syntactic (provability-code) derivation are admitted, yielding $\mathrm{Th}(D_{n+1}) \vdash \neg\gamma_n$.

Together, $\mathrm{Th}(D_{n+1}) \vdash \gamma_n$ and $\mathrm{Th}(D_{n+1}) \vdash \neg\gamma_n$ give $v_{n+1}(\gamma_n) = \{T, F\}$ by Definition 4.3. The non-explosion axiom (Axiom 4) ensures this glut does not collapse the entire L_{n+1}.

(iii) By Definition 4.2, True_n is definable in L_{n+1} and $\mathrm{True}_n(\gamma_n)$ is the sentence "γ_n is true in D_n." Since γ_n is indeed true in the standard model of D_n (it correctly asserts its own unprovability), $D_{n+1} \models \mathrm{True}_n(\gamma_n)$ and $D_{n+1} \not\models \neg\mathrm{True}_n(\gamma_n)$. Hence $v_{n+1}(\mathrm{True}_n(\gamma_n)) = \{T\}$. □

Remark 5.4. The contrast between parts (ii) and (iii) of Theorem 5.3 encapsulates the transition mechanism. The glut $v_{n+1}(\gamma_n) = \{T, F\}$ arises from the co-presence of two different notions of truth: γ_n, evaluated against the internal provability predicate Prov_n, generates a contradiction because the syntax of L_n collapses truth and provability. The Tarski jump True_n separates them, and the glut dissolves when γ_n is re-evaluated via True_n.

5.3 Fixed-Point Structure via Lawvere

Theorem 5.5 (Diagonal Fixed Point at Layer Boundary). *Let $e : D_n \to (D_n \to \mathcal{P}(\{T, F\}))$ be the Gödel numbering extended to functions. Then for every endomorphism $t : \mathcal{P}(\{T, F\}) \to \mathcal{P}(\{T, F\})$, there exists $\phi \in D_n$ such that $e(\phi)(\phi) = t(e(\phi)(\phi))$. In particular:*

- *Taking $t = \mathrm{id}$ recovers the liar-type self-referential fixed point.*
- *Taking $t(x) = \{T, F\} \setminus x$ recovers the Gödel sentence as the fixed point of complementation.*

Proof. Apply Theorem 2.4 with $A = D_n$, $Y = \mathcal{P}(\{T, F\})$, and $f = e$. The Gödel numbering provides the required point-surjection in the effective sense: every computable function $D_n \to \mathcal{P}(\{T, F\})$ is represented by some $\phi \in D_n$ under the standard recursion-theoretic encoding. The fixed-point equation $e(\phi)(\phi) = t(e(\phi)(\phi))$ is exactly Lawvere's diagonal. □

Corollary 5.6 (Contradiction as Fixed Point). *The glut $v_{n+1}(\gamma_n) = \{T, F\}$ is not an ad hoc assignment but is forced by the categorical structure: γ_n is the Lawvere fixed point of the complement endomorphism $t(x) = \{T, F\} \setminus x$ on $\mathcal{P}(\{T, F\})$.*

Proof. Immediate from Theorem 5.5 with $t(x) = \{T, F\} \setminus x$ and from Theorem 5.3(ii). □

6 Conservative Extension over ZFC

A critical structural requirement of $\mathcal{U}_{\mathrm{EML}}$ is that the paraconsistency of higher layers does not contaminate the classical base layer.

Theorem 6.1 (Conservative Extension of ZFC)**.** *For any sentence $\phi \in L_0$, $\mathcal{U}_{\mathrm{EML}} \vdash \phi \iff \mathrm{ZFC} \vdash \phi$.*

Proof. Let $\phi \in L_0$. By Axiom 1, $v(\phi) = v_0(\phi)$. The valuation v_0 is defined strictly over the classical domain $D_0 = V$ with Tarskian semantics (Axiom 2). The recursive definition of $v_{\alpha+1}$ introduces paraconsistent values only for formulas involving True_α or Prov_α, predicates that are not present in L_0 by Axiom 3. Hence for $\phi \in L_0$, the truth conditions in $\mathcal{U}_{\mathrm{EML}}$ coincide with those in classical ZFC, and no glut or gap is assigned. □

Lemma 6.2 (Non-Contamination Lemma)**.** *If $v(\phi) = \{T, F\}$, then $\phi \notin L_0$.*

Proof. Suppose $\phi \in L_0$ and $v(\phi) = \{T, F\}$. By Axiom 1, $v(\phi) = v_0(\phi)$. But v_0 maps into $\{\{T\}, \{F\}\}$ by Axiom 2, contradicting $v(\phi) = \{T, F\}$. Hence all contradiction tolerance points are confined to L_α for $\alpha \geq 1$. □

7 Transfinite Extension

7.1 Extension Through All Ordinals

Definition 7.1 (Transfinite EML Hierarchy)**.** For each ordinal α, the language L_α is the first-order language of D_α, and $v_\alpha : L_\alpha \to \mathcal{P}(\{T, F\})$ is defined by Definition 4.3 with D_n replaced by D_α. The global evaluation map v is fixed by Axiom 1.

Theorem 7.2 (Transfinite Completeness)**.** *For every $n < \omega$ and $\phi \in L_n$,*

$$v_\omega(\phi) \in \{\{T\}, \{F\}, \{T, F\}\}.$$

That is, D_ω is gap-free: every formula in the ω-layer receives at least one definite truth value.

Proof. By induction on n. *Base case $n = 0$*: classical ZFC/PA is complete over its standard model, so $v_0(\phi) \in \{\{T\}, \{F\}\} \subseteq \{\{T\}, \{F\}, \{T, F\}\}$. *Inductive step*: assume the claim for all $k \leq n$. Any $\phi \in L_{n+1} \setminus L_n$ either (a) involves True_n, in which case $v_{n+1}(\phi) \in \{\{T\}, \{F\}\}$ by Theorem 5.3(iii); or (b) is the Gödel sentence γ_n, in which case $v_{n+1}(\gamma_n) = \{T, F\}$ by Theorem 5.3(ii). In both sub-cases the value lies in $\{\{T\}, \{F\}, \{T, F\}\}$. □

Theorem 7.3 (No Gap at Limit Layers)**.** *For every limit ordinal λ and $\phi \in L_\lambda$, $v_\lambda(\phi) \neq \emptyset$.*

Proof. Since $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$, there exists $\beta < \lambda$ with $\phi \in L_\beta$. By Theorem 7.2 applied to the sub-hierarchy up to β, $v_\beta(\phi) \neq \emptyset$. Since $D_\lambda \supseteq D_\beta$, the valuation v_λ extends v_β monotonically in the FDE order, so $v_\lambda(\phi) \neq \emptyset$. □

7.2 The Open Hierarchy

Theorem 7.4 (Non-Closure of the Hierarchy). *For every ordinal α, there exists $\gamma_\alpha \in L_\alpha$ with $v_\alpha(\gamma_\alpha) = \emptyset$, forcing the generation of $D_{\alpha+1}$. In particular, the hierarchy $(D_\alpha)_{\alpha \in \mathrm{Ord}}$ has no maximal element.*

Proof. The theory $\mathrm{Th}(D_\alpha)$ extends PA for every α (since $D_0 = V$ contains a model of PA and each $D_{\alpha+1}$ conservatively extends D_α in its base language by Theorem 6.1). Hence Theorem 2.5 applies uniformly, supplying a Gödel sentence γ_α with $v_\alpha(\gamma_\alpha) = \emptyset$ at every ordinal stage. □

8 Correspondence with Known Systems

Proposition 8.1 (Embedding of da Costa Systems). *Each da Costa system C_n ($n \geq 1$) embeds into $\mathcal{U}_{\mathrm{EML}}$ via a faithful translation $\iota_n : C_n \to L_n$ preserving non-explosion.*

Proof. The da Costa system C_n is axiomatised by classical logic minus *ex contradictione*, plus consistency operators $\circ_n \phi := \neg(\phi \wedge \neg\phi)^{(n)}$. Map each formula ϕ of C_n to the corresponding formula in L_n by replacing $\circ_n$ with the predicate $v_n(\phi) \neq \{T, F\}$. Non-explosion is preserved because by Axiom 4, the FDE valuation in $\mathcal{U}_{\mathrm{EML}}$ does not derive all consequences from a glut. □

Proposition 8.2 (Embedding of Priest's LP). **LP** *embeds into the fragment (L_1, v_1) of $\mathcal{U}_{\mathrm{EML}}$ as the restriction to gap-free semantics.*

Proof. By Theorem 5.3(ii), $v_{n+1}(\gamma_n) = \{T, F\}$ (a glut); by Theorem 7.2, v_ω has no gaps. Restricting v_1 to formulas that receive no gap value $\emptyset$ gives exactly Priest's three-valued matrix $\{\{T\}, \{F\}, \{T, F\}\}$ with LP-style connectives. □

Proposition 8.3 (Subsumption of FDE). *For each finite n, the pair (L_n, v_n) is an FDE model.*

Proof. The four cases of v_n in (4) correspond exactly to the four FDE truth values, and the connective evaluation in D_n satisfies the FDE clause laws by Proposition 4.5. □

Proposition 8.4 (Subsumption of Lawvere Fixed-Point Semantics). *The Lawvere fixed-point structure (Theorem 5.5) is realised at every finite layer boundary.*

Proof. By Corollary 5.6, the Gödel sentence is the Lawvere fixed point for the complement endomorphism on $\mathcal{P}(\{T, F\})$. The Tarski jump provides the point-surjection required by Theorem 2.4. □

System	Location in $\mathcal{U}_{\text{EML}}$	Key property preserved
ZFC / PA	(L_0, v_0)	Classical bivalence
da Costa C_n	(L_n, v_n)	Consistency operators
Priest **LP**	(L_1, v_1)	No gaps, gluts allowed
FDE (Belnap–Dunn)	(L_n, v_n), all n	Four-valued matrix
Lawvere fixed-point	boundary L_n/L_{n+1}	Self-referential diagonal

Table 1: Location of known systems within $\mathcal{U}_{\text{EML}}$.

9 Main Theorem

Theorem 9.1 (Unified EML Main Theorem)**.** *The system $\mathcal{U}_{\text{EML}}$ satisfies all of the following:*

(i) ***(Paraconsistency)*** *$\mathcal{U}_{\text{EML}}$ is paraconsistent: non-explosion holds globally (Axiom 4, Definition 2.3).*

(ii) ***(ZFC-Compatibility)*** *The base layer (L_0, v_0) is classically bivalent and conservative over* ZFC *(Axiom 2, Theorem 6.1).*

(iii) ***(Layer Isolation)*** *Higher-layer truth predicates cannot alter valuations fixed at lower layers (Axiom 1, Lemma 6.2).*

(iv) ***(Fixed-Point Contradiction Structure)*** *At every finite boundary L_n/L_{n+1}, the Gödel sentence γ_n is the Lawvere fixed point of complementation on $\mathcal{P}(\{T, F\})$ and satisfies $v_{n+1}(\gamma_n) = \{T, F\}$ (Theorem 5.3, Corollary 5.6).*

(v) ***(Glut Resolution via Tarski Jump)*** *The glut at each boundary is resolved in the same layer via the external truth predicate: $v_{n+1}(\text{True}_n(\gamma_n)) = \{T\}$ (Theorem 5.3(iii)).*

(vi) ***(Transfinite Openness)*** *The hierarchy $(D_\alpha)_{\alpha \in \text{Ord}}$ is strictly increasing in expressive power, with no maximal layer (Theorem 7.4).*

(vii) ***(Gaplessness at Limit Stages)*** *For every limit ordinal λ, the layer D_λ is gap-free: $v_\lambda(\phi) \neq \emptyset$ for all $\phi \in L_\lambda$ (Theorem 7.3).*

Proof. Parts (i)–(vii) collect the results cited parenthetically above; each is established in the referenced theorem, axiom, or lemma. □

10 Philosophical Remarks

Remark 10.1 (On the Nature of Layer Transcendence). The hierarchy (D_α) resembles but is not identical to the cumulative set-theoretic hierarchy $V = \bigcup V_\alpha$. In V, the jump $V_\alpha \to V_{\alpha+1}$ is driven by power-set formation; in $\mathcal{U}_{\text{EML}}$, it is driven by *semantic inadequacy*—the inability of a layer to assign a truth value to its own Gödel sentence. The paraconsistent glut $\{T, F\}$ at the boundary is not a pathology to be resolved but the formal marker of the boundary itself.

Remark 10.2 (Relationship to Dialethism). Priest's dialetheism [7] holds that some contradictions are genuinely true, i.e., that some sentences ϕ satisfy $v(\phi) = \{T, F\}$. $\mathcal{U}_{\text{EML}}$ is not committed to dialethism in the metaphysical sense: the gluts appearing in $\mathcal{U}_{\text{EML}}$ are *epistemically layer-relativised*. By Theorem 5.3(iii), the glut $v_{n+1}(\gamma_n) = \{T, F\}$ dissolves upon ascent via the Tarski jump, and γ_n acquires the classical value $\{T\}$ when evaluated under True_n. The contradiction is a transitional artifact, not a permanent feature of reality.

Remark 10.3 (Directions for Future Work). Three extensions suggest themselves. First, a fully categorical formulation—replacing layers with locally cartesian closed categories and the Tarski jump with a hypercompletion functor—would connect $\mathcal{U}_{\text{EML}}$ to Aczel's non-well-founded sets [1] and Barwise–Etchemendy's situation theory. Second, the relationship between the transfinite EML hierarchy and Woodin's Ω-logic [10] merits investigation. Third, the computational interpretation of layer transitions—via realizability or game semantics—may yield a proof-theoretically constructive account of the paraconsistent boundary values.

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